Optimal surface salinity perturbations of the Meridional Overturning Circulation

Florian Sévellec (Yale Univ.),
Thierry Huck (CNRS), Jérôme Vialard (IRD)
and Alexey Fedorov (Yale Univ.)

Department of Geology and Geophysics,
Yale University

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Climate context (1) : The MOC

Slow dynamics of the ocean:

Meridional Overturning Circulation (MOC)
- Intensity of $\sim 18$ Sv
- Time scale of $\sim 500$ ans

Northward transport of heat influencing the European climate
$\Rightarrow$ Variability of the meridional overturning circulation
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Increase of precipitation in the north Atlantic

→ Josey and Marsh (2005)
Climate context (2) : North Atl. P-E

- Increase of precipitation in the north Atlantic
  → NCEP reanalysis

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What is the impact of the SSS modification on the meridional overturning circulation?
Approach

⇒ Forced variability of the ocean circulation

- **Linear approach**: weak variations (perturbations) of the ocean circulation
- Generalized stability analysis:
  - Atmosphere ⇒ optimal initial and stochastic perturbation (Farrell and Ioannou, 1996)
  - Ocean ⇒ optimal initial perturbation (Moore and Farrell, 1993)
  - 3 box THC ⇒ optimal initial and stochastic perturbation (Tziperman and Ioannou, 2002)
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Maximization method: Oceanic circulation application

Goal:

- Optimal impact of the SSS on the circulation

Lagrange parameters method

- Functions to maximize $\langle F|u(t)\rangle$ (or $\langle F|u(t)\rangle^2$):
  - Meridional Overturning Circulation (MOC) at the latitude and depth of its steady state maximum (or its variance)

- Constraints
  1. Normalisation: $\langle u(0)|S|u(0)\rangle = 1$
  2. Salt conservation: $\langle C|u(0)\rangle = 0$
  3. Only surface salinity perturbation: $|u(0)\rangle = P\,|u'\rangle$
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  - Methodological study

- Planetary geostrophic model:
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Upper bound in the 2D model

- **Initial SSS perturbation**: Great Salinity Anomalies (GSA, Belkin et al., 1998)
  0.5 psu on 250 m $\Rightarrow$ 2 Sv

- **Constant FW perturbation**: Hydrological cycle modification in the global warming scenario (Held and Soden, 2006)
  4% (3 cm yr$^{-1}$) $\Rightarrow$ 0.14 Sv

- **Stochastic FW perturbation**: Using 2 different reanalysis datasets
  5 cm yr$^{-1}$ $\Rightarrow$ 4.6 Sv
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Results in the PG model

- **Variability:**
  - Large scale gradient SSS *efficiently stimulates* a North Atl. multidecadal oscillation.

- **Surface boundary condition:**
  - The sensitivity pattern weakly depends on the surface boundary condition.
  - The intensity of the response strongly depends on the boundary condition.
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Application in an Ocean General Circulation Model

Model: OPA 8.2, ORCA2, OPATAM

- MAX(MOC) = 7 Sv (48°N)
- MAX(MHT) = 0.6 PW (27°N)

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Optimal surface salinity perturbations of the MOC
Optimal initial SSS perturbation for the MOC

- Maximum growth after 10.5 yr
Introduction: Climate context

Optimal SSS perturbation of the MOC

Conclusions

Approach

Application in an Ocean General Circulation Model

\[ \nu'_\text{surf} > 0 \]

\[ |\beta \partial_\theta S'| \gg |\alpha \partial_\theta T'| \]

\[ |\alpha \partial_\phi T'| \gg |\beta \partial_\phi S'| \]

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Optimal surface salinity perturbations of the MOC
Finite time growth mechanism

\[ \alpha \partial_\phi \bar{T} \gg \beta \partial_\phi \bar{S} \]

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Finite time growth mechanism

\[ \alpha \partial_{\phi} \bar{T} \gg \beta \partial_{\phi} \bar{S} \]

\[ \text{MEAN DENSITY (kg m}^{-3}\text{), LAT = 46°N, Z-MEAN = 0 – 612 m} \]

\[ \text{MEAN TEMPERATURE (°C)} \]

\[ \text{MEAN SALINITY (psu)} \]

\[ \text{LONGITUDE (°)} \]

\[ \text{LATITUDE} \]

\[ \text{SURFACE PLAN} \]

\[ \text{SSS’} > 0 \]

\[ \text{T’} \Rightarrow \Delta T’ > 0 \]

\[ \text{u’} > 0 \]

\[ \text{v’} > 0 \]

\[ \text{SSS’_{north}} > 0 \Rightarrow \nu’_{surf} > 0 \]
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\[ \alpha \partial_\phi \bar{T} \gg \beta \partial_\phi \bar{S} \]

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Nonlinear - linear comparison

- Relative error: less than 20%
- Max bound: GSA $\Rightarrow 0.75$ Sv (11% of MOC)
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Results

- **Efficient method** to obtain the optimal initial perturbation:
  - ⇒ **Explicit solution** (adj. model)

- Results of the 2D, PG and OGCM models
  - ⇒ Similarity:
    - In the 2D model, the sensitivity is dominated by the salinity, and the response is dominated by the temperature.
    - ⇒ Difference: Transient growth mechanism

- Optimal SSS perturbation of the MOC in an OGCM
  - ⇒ Growth mechanism

- Upper bound of the impact of SSS on MOC
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Future work

- **Optimal wind stress perturbation**
  - Impact of the Southern Ocean
  - Mechanism of the finite time growth
- Seasonal cycle (non-autonomous operator)
  - Sensitivity to the season
- Tropical study:
  - Optimal ocean perturbation and phase locking of ENSO (ENSEMBLES, European project for climate changes prediction)
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Thank you for your attention
Introduction: Climate context
Optimal SSS perturbation of the MOC
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Optimal surface salinity perturbations of the MOC
Optimal initial SSS perturbation

Perturbation evolution (autonomous problem):

\[ \partial_t |u\rangle = A |u\rangle, \]

\[ \Rightarrow |u(\tau)\rangle = M(\tau) |u(0)\rangle = e^{A\tau} |u(0)\rangle. \]

Explicit solution (using the adjoint model) of the optimal initial perturbation:

\[ \Rightarrow |u(0)\rangle = P |u'\rangle \]

\[ |u'\rangle = (2\gamma_1)^{-1} \left( N^{-1} P^\dagger M^\dagger(\tau) |F\rangle - \gamma_2 N^{-1} P^\dagger |C\rangle \right), \]

with \( N = P^\dagger SP \),

\[ \gamma_1 = \text{fct} \left( M^\dagger(\tau) |F\rangle, |C\rangle, N, P, \gamma_2 \right) \quad \text{and} \]

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\[ \Rightarrow \text{Solution depends on the maximization delay } \tau \]
Efficient method:

Maximization under constraints: \( dG(\gamma, |u_0\rangle) = 0 \)

- Measure: Linear function

\[
G(\gamma, |u_0\rangle) = \langle F | M(\tau) | u_0 \rangle - \gamma (\langle u_0 | S | u_0 \rangle - 1)
\]

Explicit solution:

\[
|u_0\rangle = \pm \frac{S^{-1}M^\dagger(\tau) |F\rangle}{\sqrt{\langle F | M(\tau) S^{-1} M^\dagger(\tau) | F \rangle}}
\]

- Measure: quadratic norm

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G(\gamma, |u_0\rangle) = \langle u_0 | M^\dagger(\tau) S_2 M(\tau) | u_0 \rangle - \gamma (\langle u_0 | S_1 | u_0 \rangle - 1)
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Eigenvalue solution:

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