NewWaves, Solitons and Spreading

Paul H. Taylor$^1$ and Christopher Swan$^2$

$^1$Department of Engineering Science, University of Oxford, Parks Road, Oxford OX1 3PJ, U.K.
$^2$Department of Civil and Environmental Engineering, Imperial College, London SW7 2UB, U.K.

Abstract. The simplest model for random waves is based on Gaussian statistics. The idea of an average shape for an extreme wave crest in a linear sea, due to Lindgren and subsequently known as NewWave, is used as a starting point for a review of the effects of non-linearity in the production of extreme waves in random seas. Both experiments and numerical simulations show that the physics of the evolution of directionally spread wave fields is different to that of unidirectional waves. Thus, efforts to explain the occurrence of rogue waves should reflect the wave spreading obvious in nature.

1 Introduction

What is a rogue wave? A possible definition might be

“An extreme wave event which apparently occurs on average more often than would be predicted using the tail of the Rayleigh distribution for statistics of linear waves - even after allowing for the obvious 2nd order crest-trough asymmetry of steep waves”.

Let us start by assuming that the possible occurrence of rogue waves can be explored by means of solutions to the full Euler equations of potential flow for an ideal incompressible and constant density fluid. There is no dissipation or energy input into the fluid domain. Hence, the fluid remains irrotational. Further, we shall assume that there are no ocean currents and that the water is deep. The free-surface boundary conditions neglect any influence of the air above the surface and surface tension is ignored. Since our models cannot calculate beyond wave breaking, we further assume that wave overturning does not occur and that white water is not important. Although these assumptions represent an enormous simplification of reality in the open ocean during severe storms, the prediction of the evolution of surface water waves still remains challenging.

If rogue waves exist, they must be inherently non-linear. If they are defined to be more severe than a 2nd order model would predict, then they must arise from 3rd and
higher order interactions – if such a perturbation expansion is meaningful. This contribution attempts a partial, somewhat personally biased review of the literature on steep wave focussing.

2 Second Order Models and NewWave

A Stokes 2nd order model for steep waves represents the 1st non-linear term in a perturbation expansion for surface waves in terms of wave steepness. As such, it can only weakly be non-linear – the kinematics at the free surface are different for crests and troughs and there is the radiation-stress driven return flow beneath localized wave steep groups. However, the underlying dynamics are still assumed to be linear – the position of the dominant linear Fourier components for any time can be predicted using simple linear dispersion.

So long as only a 2nd order accurate model is used, then the properties of extremes in an underlying effectively linear random Gaussian process are simple to describe. The average shape of a tall crest tends to the scaled auto-correlation function, Lindgren [1]. Tromans [2] introduced this idea into offshore engineering where it has become known as NewWave. One of the reasons why this is an attractive model is that it connects the (averaged) properties of the largest extremes in a sea-state to the properties of all the waves in that sea-state. The auto-correlation function is simply the Fourier transform of the power spectrum. This model is convenient for both physical experiments and computation as it gives a localized wave group consistent with the broad-banded and directional spread nature of real sea-states.

Second order corrections to an underlying linear model are consistent with much field data. The average shapes of large deep-water waves measured during severe winter storms at Tern, a northern North Sea platform in 170m of water, are entirely consistent with the NewWave model [3]. Both the simple case of a single point measurement of surface elevation in time and the more complex case of the simultaneous time history at a second location given a wave crest of given size at an adjacent point were studied. The prediction of this latter case required directional sea-state information obtained from a directional wave-rider buoy. For small waves, the full solution by Lindgren [1] for the average shape of waves is required. However, in the limit of a large event ($A/\sigma > 2$, where $A$ is the individual wave amplitude and $\sigma$ is the rms surface elevation), this exact Lindgren solution tends to the shape of the auto-correlation function. As well as being a good model for the average shape of large wave crests on deep water, analysis of wave data measured in 17m water depth during the recent WACSIS joint industry project confirms the validity of this approach for steep waves on intermediate water depth in winter storms.

At second order, all the terms quadratic in wave amplitude are slaved to the (assumed) underlying linear components which move in a manner consistent with simple linear dispersion. Thus, brute force simulation of random sea-states can provide benchmark statistics on crest elevations etc. (Forristall [4], Prevosto in this workshop). A more elegant method of obtaining the same short-term wave statistics is the re-
response surface (FORM) approach of Tromans. This approach has been applied to the statistics of crest elevation for deep-water uni-directional waves [5].

The apparent consistency between field data in severe storms and 2nd order models implies that rogues are rare – perhaps requiring a special set of circumstances that we don’t yet understand.

3 Beyond Second Order for Wave Evolution

Theoretical models of steep waves beyond 2nd order in wave steepness must account for non-linear dispersion if they are to be accurate over long distance and time scales. The simplest example of non-linear dispersion is the amplitude dependent Stokes correction to the phase speed of regular waves arising at 3rd order in the theory.

The idea of having shorter, slower waves overtaken by longer, faster ones - frequency-based focussing - pre-dates the derivation of the non-linear Schrödinger equation and the derivation of a general solution method by inverse scattering, Zakharov and Shabat [6]. The interaction of linear dispersion and wave non-linearity was first studied by Lighthill [7] and is briefly discussed in his book [8], p.462.

At 3rd order for narrow-banded wave groups, non-linear evolution equations (NLEEs) can be used to explore the effects of wave amplitude in the dynamics. The consequences are profound – and different for group structure in a longitudinal down-wave direction or structure laterally along the wave crests. The simplest NLEE is the non-linear Schrödinger equation, Zakharov and Shabat [6], Yuen and Lake [9] and Peregrine [10]. For isolated uni-directional wave groups on deep water described by the positive (NLS+) version of the equation, there are soliton solutions where linear dispersion is balanced by 3rd order amplitude dispersion to produce permanent and robust localized groups.

Both one-dimensional physical experiments, Baldock, Swan and Taylor [11], and numerical and analytic modelling [12] show that non-linearity co-operates with linear focussing, when an extreme event is produced by having long wave components overtake short ones. The following simple argument, originally due to Lighthill, shows how this co-operation can occur. Consider a wave group with shorter waves ahead and longer ones behind. Due to linear dispersion the longer waves will slowly catch up with the shorter ones. In the centre of the group the waves are higher than those at the edges. Since high waves move faster than small waves of the same wavelength, the waves in the centre of the group will catch up with those ahead – amplitude dispersion. The waves ahead will be compressed, the local wavelength is reduced, and those behind stretched, leading to further linear dispersive changes to the focus event. A simple approximation for this physical process is captured by the NLS equation.

Although the NLS equation is the lowest order NLEE for deep-water waves, it is not quantitatively accurate for isolated wave groups. Dysthe [13], Stiassnie [14] and Lo and Mei [15] include other terms to improve both the dispersive and non-linear aspects of the physics.
Probably the most important piece of the physics not captured by the NLS equation is the dynamical consequence of the return flow beneath the wave group. Any finite size deep-water wave transports fluid forwards in the direction of wave advance – Stokes drift. However, an isolated wave group rides on a quiescent ocean. Thus, there must be a continuous return flow beneath the wave group balancing the Stokes drift. The length-scales associated with this flow are those of the whole group, not those of individual waves. In a perturbation expansion for a wave group, this return flow occurs at 2nd order and is associated with the set-down beneath the group. To this order there is no effect on the wave dynamics. However, there is a physical effect at 3rd order. The large waves in the centre of the group ride on a locally opposing current – the return flow. Thus, the full dynamics to 3rd order are weaker than predicted by the NLS equation. The relative importance of the return flow depends on the length of the group. If the group is compact (the spectrum broad-banded), this effect is important.

In an engineering context, the direction and magnitude of this return flow is important, both for waves on deep and shallow water. Being backwards in direction, the horizontal fluid velocity subtracts from the usual in-line kinematics beneath the wave crests but adds to the backward flow beneath wave troughs. Thus, for steep waves the magnitude of the net horizontal kinematics at elevations below the level of the deepest trough can be larger beneath a deep trough than beneath tall crests. Of course, the highest kinematics occur in the in-line direction within the crests above mean-sea-level.

One result, clear from experiment [11], is the fast rate at which non-linear energy transfer between spectral components can occur in uni-directional wave groups. Starting 10 periods before focus, a wave group could become $O(25\%)$ higher than predicted by linear theory at focus, even after allowing for 2nd order effects. Such a localized wave group is formed which is more compact, taller and longer lasting than would be predicted by simple linear theory (even if 2nd order corrected). Indeed, if the spectral components in the converging group are chosen suitably, a group is formed which at least locally closely approximates a solitary wave group. Perhaps such a soliton might be a good candidate for a model of rogue waves.

The discussion thus far has concentrated on isolated wave groups on quiescent water. In reality, every large event in a random sea emerges and presumably disperses back into a random background. Does the presence of this background have any effect on the coalescence properties of a large wave event? Yuen and Lake [9] discuss the work of Albers and Saffman on this problem. For sufficiently narrow-banded problems so that the NLS equation is valid, they conclude that the modulational (Benjamin-Feir) instability of a Stokes wave train is weakened and can be eliminated if the background is sufficiently strong. This can be understood as phase randomization of the 3rd order wave-wave interactions (which feed back to the principal dynamics only when the 3rd order difference terms remain in phase with the otherwise dominant linear components). Both recent experiments by Stansberg [16] in a long wave tank and full simulations of the Euler equations by Mori and Yasuda (this workshop) show that the statistical properties of a uni-directional random wave field do change with distance and or time. Hence, some cumulative non-linear effects survive randomization.
In contrast to 1-D waves with group structure along the mean wave direction but crests running transversely to infinity, variation in the height of a wave group along the wave crest direction leads to the NLS− or defocusing version of the equation. Non-linearity is still important but the focussing of components to produce an extreme event is weakened—the group is now lower but longer-crested. Consistent with this, there are now no soliton solutions propagating in isolation. Thus, group structure in the mean wave and transverse directions has different consequences for the production of extreme events.

4 Wave Focussing for Directionally Spread Components

Real waves on the open ocean are directionally spread. The wave crests do not run at constant height from horizon to horizon. Instead, a strongly two-dimensional pattern is obvious. A simple argument indicates that non-linearity could well be less important for a spread sea than it is for the 1-D case. A wave group can be built up from many individual Fourier components, each with its own phase speed and group velocity. The group velocity is defined as the speed of propagation of energy of that component. Due to the range of wavelengths present, the length of the group will increase with time away from the instant of focus when all the components are in phase. Thus, the length of the group increases asymptotically with time as \( L \sim O(t) \). As the group lengthens, the wave height drops to satisfy conservation of total energy. The total energy of the group is constant as \( E \sim O(A^2L) \), where \( A \) is the amplitude of a linearly dispersing wave group. Thus, the amplitude decays asymptotically as \( A \sim O(t^{-1/2}) \) as the energy becomes more and more spread out along a line. This simple asymptotic behaviour is consistent with the linear part of the NLS equation. Now, the NLS equation contains both linear and cubic terms in the wave amplitude. The net effect of the cubic term in the NLS equation on the amplitude over long times is \( O(A^3t) \sim O(A) \), assuming the effect is cumulative. Thus, there are long-term consequences for the evolution of the group in 1-D.

The situation is quite different for directionally spread wave groups. A localized wave group now disperses out over a plane rather than along a line. The wave amplitude decays asymptotically as \( t^{-1} \). Over long times, the long-term contribution to the amplitude from the cubic NLS term is \( O(A^3t) \sim O(A^2) \), being negligible compared to linear behaviour which is \( O(A) \). Thus, wave-wave interactions are more localized—needing a large wave group close to the linear focus to produce significant effects.

Both versions (NLS+ and NLS−) of the Schrödinger equation in 1-D are solvable by inverse-scattering techniques. The combined spread sea \((x,y,t)\) version is not. The spread sea \((x,y,t)\) version of the NLS equation also suffers from a catastrophic defect—energy is predicted to leak to higher and higher wavenumber. This leakage is non-physical and implies that long-time simulations in 2D using the NLS equation would be severely flawed. In an exciting development described at this workshop, Trulsen presented an improved version of the wave evolution equation with this en-
ergy leakage problem cured. This work may represent an elegant starting point to study the statistics of the freak wave problem with a realistic amount of computer resources. However, benchmarking against fully non-linear schemes will be required as the range of free wavelengths that any evolution equation can simulate is restricted and real ocean wave spectra are broad-banded. Note also that NLEEs are usually studied numerically with pseudo-spectral numerical schemes, with the computational effort scaling as $O(N \log N)$, where $N$ is the number of points on the surface.

As discussed above, the standard NLS equation does not include the dynamic effects of the return current beneath the wave group, which is driven by the Stokes drift. For a directional spread wave group, the return flow still exists, as the Stokes drift is still transporting fluid forwards to the lead edge of the group. However, this flow is now locally considerably weaker - it is able to spread out sideways horizontally as well as penetrating downwards underneath the centre of the group. Thus, the backward advection velocity of the tall waves in the centre of the group is significantly reduced, and the effect on the wave dynamics at 3rd order is smaller. There is a NLEE with this effect included – the Davey-Stewartson equation. For deep water spread seas, the Davey-Stewartson equation is not solvable by inverse scattering, Ablowitz and Clarkson [17].

Overall, we have two opposing effects – the directionally spread groups are compact and tall for a shorter time than uni-directional groups, simply due to linear dispersion. This reduces the time over which 3rd order non-linear effects are able to act. However, the dynamical consequences of the return flow (opposing non-linear focusing) are also weaker. Further, it is known that new physics enters with the extra spatial dimension – the resonant interactions of Phillips [18] and Hasselmann [19], which permanently transfer energy from 3 components to a 4th.

All of this suggests that the behaviour of directional spread wave groups is likely to be both qualitatively as well as quantitatively different to uni-directional wave groups. Thus, it is probably inappropriate to base the analysis of field data on one-dimensional models. However, this discussion has been based on known properties of solutions to the non-linear evolution equations. Whether, these approximate equations capture enough of the behaviour of solutions to the full Euler equations for water waves remains to be seen.

5 Fully Non-Linear Simulations

Many different computational approaches have been proposed for the solution of the full water wave equations. For uni-directional problems perhaps the best is the boundary integral technique used by Dold [20] and others.

For directionally spread waves, there are several pseudo-spectral methods in the literature. These are attractive as the main computational task is taking Fourier transforms, for which efficient FFT algorithms exist. Thus, the computational effort scales as $O(N \log N)$, where $N$ is the number of points on the surface. This is the same scaling, albeit with a larger coefficient in front, as nonlinear evolution equations – which
are considerably more restricted in the range of wavelengths that can be accurately represented. Although Fenton and Rienecker presented the first full non-linear Fourier scheme [21], a more recent one devised by Dommermuth and Yue [22] is widely used. In work presented at this workshop, Mori and Yasuda used it in their study of the statistics of random realizations of sea-states. They conclude that 3rd order interactions are important both for uni-directional and directionally spread waves.

A new, efficient and robust numerical scheme for the solution for the full Euler equations for water waves has recently been derived [23]. This is based on the Dirichlet-Neumann (G-) operator of Craig and Sulem [24]. This operator permits the accurate conversion of the velocity potential on the free surface into the normal gradient of this velocity potential, even with a large range of wavenumber components represented. This step is essential in any time-integration scheme for water waves based on the exact dynamic and kinematic boundary conditions. In common with all pseudo-spectral schemes, this approach is restricted to non-overturning waves: the surface elevation is assumed to be a single-valued function of the horizontal coordinates. Thus, only the early stages of wave breaking can be studied. However, the main advantages of this new scheme are its efficiency and robustness. It can be run on a PC and is sufficiently robust that it can be used to predict the time histories and local kinematics of near breaking waves. The results are in excellent agreement with the high quality wave basin data recently reported by Johannessen and Swan [25] and discussed at this workshop by Swan in a separate contribution.

6 Results from Steep Wave Simulations

A fundamental difficulty with numerical modelling is the choice of suitable initial conditions to start the simulation. With random initial conditions, considerable computer resources are required to represent a large enough patch of ocean surface evolving over a long enough time to generate useful statistics. The average shape of an extreme wave crest helps here. This defines an isolated NewWave group, which in some sense is typical of the occurrence of large events in a random sea-state. Away from the focus point, such a localized wave group would disperse. If the group is isolated, it arises from and presumably disperses back to a linear background state. Such a NewWave group well before the focus time provides a suitable initial condition for numerical simulation. The non-linear physics of steep waves can then be explored with a ‘clean’ calculation. At a later stage, the presence of the random background could be included.

Although numerical work is still in progress, examples of fully non-linear focussed wave groups on deep water show that the introduction of a realistic degree of directional spreading has dramatic consequences for the focussing of a steep wave group [26]. Using a realistic wave spectrum and directional spreading to define a NewWave well before focus, Bateman at Imperial College in London has followed the fully nonlinear evolution in time. The wave-wave interactions, which are so important for uni-directional wave groups, are considerably modified by directional spreading. In
particular, the peak surface elevations arising in directional spread focussed wave groups are similar to those predicted by simple 2\textsuperscript{nd} order theory if the spreading is as large as that commonly observed in severe winter storm waves. However, the underlying dynamics of spread group focussing are not close to linear – in the vicinity of the focus event, the group becomes more compact in the mean wave direction but wider in the along-crest direction and the position of the occurrence of the highest surface elevation is shifted. Thus, the shape of the wave group is modified – and for engineering applications the local factor on in-line wave kinematics to allow for spreading would be closer to unity than expected from linear theory. Furthermore, this localization of the total energy into a small area of sea for a relatively short time has a permanent effect even as the group subsequently diverges to infinity - the directional spreading of the group far downstream is different to that far upstream.

One clear result from the numerical runs is that the same net directionality changes arise whether the input groups are designed to create a tall crest or inverted to give a deep trough. This symmetry property is obeyed even for waves only a few per cent lower than the breaking limit for that spectral shape. Thus, only the shape and amplitude of the group are important, not the relative phasing of the ‘wiggles’ within the group. This observation implies that these large-scale group changes can only be only consistent with odd order non-linear interactions. Presumably these global changes are examples of the 3\textsuperscript{rd} order resonant interactions of Phillips [18] and Hasselmann [19]. To date, this part of the physics has only been incorporated into wavefield prediction and hindcast via perturbation interaction equations [27]. The adequacy of these interaction equations is yet to be assessed for events typical of extremes in a random sea.

7 Conclusion

It is apparent that most waves in a random sea behave for most of the time in a manner consistent with simple 2\textsuperscript{nd} order models based on linear dynamics. As a consequence, the average shape of a large wave crest, the scaled auto-correlation function NewWave, becomes a useful model. It captures considerable information about the overall statistical properties of the sea-state into a single isolated wave group.

Beyond 2\textsuperscript{nd} there is interesting non-linear dynamics to be explored. Both experiments and numerical simulations show that the evolution of directionally spread wave fields is qualitatively different to that of uni-directional waves. The important role played by soliton-type non-linear wave groups in uni-directional wave evolution seems to be absent from the physics of directionally spread seas. Thus, efforts to explain the occurrence of rogue waves should reflect the directional spreading of waves obvious in nature.

Although the possible implications of wave dynamical non-linearity on the likelihood of the occurrence of rogue waves is still not resolved, the study of steep wave groups arising by frequency focussing appears to be a useful area of research. Even if the true explanation for rogue waves lies elsewhere, improved models for the statistics
of wave crests and the associated wave kinematics would be an important contribution to oceanography and offshore engineering.

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References