A rapid fully nonlinear method in three dimensions with simulations of steep wave event.

D. Fructus and J. Grue

Mechanics Divison, Department of Mathematics, University of Oslo, Norway

Introduction

An efficient numerical scheme for simulations of fully nonlinear non-breaking surface water waves is presented. The water depth is either finite or infinite and the bottom is allowed to be varying in time and space. The method is based on a fast, rapidly converging, iterative algorithm to compute the Dirichlet to Neumann operator. This is evaluated by expanding the operator as a sum of global convolution terms and local integrals with kernels that decay quickly in space. The global terms are computed very quickly via FFT. The local terms are evaluated by numerical integration. Analytical integration of the linear part of the prognostic equations is combined with a special step size control technique. This yields a very stable and accurate time marching procedure. Zeros-padding in the spectral space represents the anti-aliasing strategy. The method requires no smoothing. The scheme is stable and accurate, even for very long time simulations of very steep wave events. The scheme is easily parallellizable. Illustration through examples are considered. In a first simulation, the bottom is varying in time. In a second one, where the water depth is infinite, the focus is put uppon long time evolution of wave group leading to the generation of rogue wave.

Fully nonlinear numerical model

We consider three-dimensional irrotational wave motion at the surface of a homogeneous incompressible fluid over a horizontal impermeable bottom.

$\mathbf{x}=(x,y)$, z, denotes horizontal and vertical coordinates. $z=0$, $z=-h$ and $z=\eta(\mathbf{x},t)$ are, respectively, the equations of the still level, of the impermeable bottom and of the free surface. Let also $\vec{\mathrm{v}}=(\mathbf{u},w)$ be the velocity field, where $\mathbf{u}=(u,v)$ and w are the horizontal and vertical velocities, so that $\vec{\mathrm{v}}=\mathrm{grad}\,\phi$. $\mathbf{u}=\nabla\phi$ and $w=\phi_z$; ϕ being the velocity potential and ∇ being the horizontal gradient. We denote with 'tildes' the quantities at the free surface, e.g. $\tilde{\phi}(\mathbf{x},t)=\phi(\mathbf{x},z=\eta(\mathbf{x},t),t)$. At the free surface, $\tilde{\mathbf{u}}$ and $\tilde{w}$ are expressed by,

$$\tilde{\mathbf{u}}=\frac{\nabla\tilde{\phi}\,-\,V\,\nabla\eta\,+\,\left(\nabla\eta\times\nabla\tilde{\phi}\right)\times\nabla\eta}{1\,+\,|\nabla\eta|^2},\qquad \tilde{w}=\frac{V\,+\,\nabla\eta\cdot\nabla\tilde{\phi}}{1\,+\,|\nabla\eta|^2},$$

$$V=\phi_n\sqrt{1+|\nabla\eta|^2}$$

The kinematic and dynamic conditions give at the surface

$$\eta_t-V\;=\;0,\qquad \tilde{\phi}_t\,+\,g\eta\,+\,\tilde{\mathbf{u}}\cdot\nabla\tilde{\phi}-\,\tilde{w}\,V\;=\;0.$$

The Laplace equation (resulting from incompressibility and irrotationality), together with the bottom impermeability, is solved exactly by means of a Green function and the method of images. For simplicity, we present here the deep water case only (h=const.=

∞). The generalization to an arbitrary bottom (allowed to be varying in time and space) will be as well presented.

The Fourier transform of V, $\mathcal{F}(V) = \hat{V}$ is then decomposed into

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + \hat{V}_3 + \hat{V}_4.$$

The equation is inverted to give (in deep water):

$$\begin{aligned}
\hat{V}_1 &= k\,\hat{\phi}, \\
\hat{V}_2 &= -k\,\mathcal{F}\{\,\eta\,V_1\,\} - \mathrm{i}\mathbf{k}\cdot\mathcal{F}\left\{\,\eta\,\nabla\tilde{\phi}\,\right\}, \\
2\pi\hat{V}_3 &= k\,\mathcal{F}\left\{\,\int\tilde{\phi}'\left[\,1-(1+D^2)^{-\frac{3}{2}}\,\right]\nabla'\cdot[\,(\eta'-\eta)\,\nabla' R^{-1}]\;\mathrm{d}\mathbf{x}'\right\}, \\
2\pi\hat{V}_4 &= -\pi k\,\mathcal{F}\{\,\eta^2\,\mathcal{F}^{-1}\{k\,\mathcal{F}\{V\}\} - 2\eta\,\mathcal{F}^{-1}\{k\,\mathcal{F}\{\eta V\}\} + \mathcal{F}^{-1}\{k\,\mathcal{F}\{\eta^2 V\}\}\,\} \\
&\quad + k\,\mathcal{F}\left\{\,\int V'\,R^{-1}\left[\,1-D^2-(1+D^2)^{-\frac{1}{2}}\,\right]\;\mathrm{d}\mathbf{x}'\,\right\}.
\end{aligned}$$

Where $R = |\mathbf{x}'-\mathbf{x}|$ and $D = (\eta(\mathbf{x}',t)-\eta(\mathbf{x},t))/R = (\eta'-\eta)/R$. The kernels of the inner integrals of $\hat{V}_3$ and $\hat{V}_4$ decay like R^{-4} and R^{-5}, respectively. These integrals are evaluated over a very limited region of the $\mathbf{x}$-plane. V_4 is determined implicitly and hence computed iteratively, for practical computations one iteration is sufficient. The method is shown to be very fast, accurate and stable without requiring any smoothing.

Applications

We present two examples of application of the method. The first simulation illustrates the method with an arbitrary bottom which is varying in time and space. A sudden collapse of the bottom generates waves that propagates at the surface of the fluid, this is illustrated in figure 1-a. A second simulation presents the evolution in infinite depth of a modulated wave. Modulational instability build up to generate a rogue wave. This is examplified in figure 1-b.

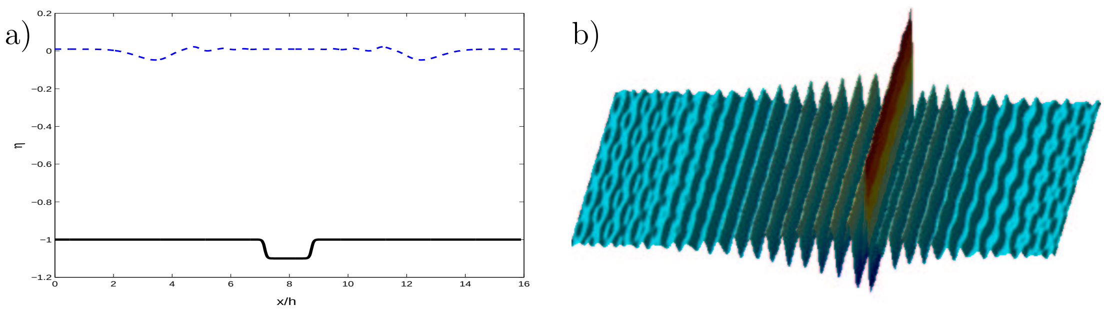

Figure 1: a): Wave elevation (- -) resulting from a sudden collapse of the bottom (–). b): Fully nonlinear simulation of a rogue wave event with the present model.

References

Clamond and J. Grue, *J. Fluid Mech.*, 2001, **447**, 337.

J. Grue, *Applied Ocean Res.*, 2002, **24**, 261-274.

D. Fructus, D., Clamond, J., Grue & Ø. Kristiansen, 2004, sub. to *J. Comp. Phys.*