The role of resonant wave interactions in the evolution of extreme wave events

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Abstract. This paper is concerned with the formation of large waves in a realistic, directionally-spread, wave-field. In particular, the role of *resonant* interactions, capable of rapidly altering the underlying wave-spectrum, is ascertained. This is investigated through the use of a fully-nonlinear numerical wave-model and Zakharov's evolution-equation. The former allows the full-nonlinearity of the wave-field to be considered, whilst the latter enables the physical mechanisms responsible for the formation of the largest waves to be determined. The paper shows that in unidirectional sea-states the *resonant* interactions enable the formation of waves that are very much larger than would be predicted by a second-order *bound* wave solution. However, it is also shown that in broad-banded directionally-spread sea-states the effect of these interactions is linked to the phase relationship between the wave components; with significant spectral evolution only occurring when the wave components do not fully focus at one point in space and time. This suggests that the focussing of wave components is not a mechanism by which so-called *freak*, or *rogue*, waves can form in a broad-banded directional wave-field. However, large increases in crest elevation are obtained in a more narrow-banded directionally-spread sea-state, characterised by a Gaussian spectrum, and representative of swell wave conditions. The present results suggest that it is in these conditions that *freak* waves are most likely to occur.

1 Introduction

The water surface elevations arising in a realistic ocean environment are commonly modelled by the sum of wave components, of different frequency, travelling in different directions. The constructive interference, or focussing, of these components at one point in space and time results in a large wave-event that is often higher and steeper than linear, or second-order, *bound* wave theory would predict. This is the result of the rapid evolution of the wave spectrum due to *resonant* interactions. As a result, the coupling of linear dispersion with nonlinear *resonant* interactions offers a possible explanation for the formation of *freak* or *rogue* waves in deep water. This paper is concerned with the evolution of realistic wave spectra. In particular, it is concerned with the relationship between the phasing of the wave components, the evolution of the wave spectrum and the resulting nonlinear crest elevation. This has been undertaken by applying two nonlinear wave-models: Bateman *et al.* (2001) and Zakharov (1968). This paper begins in §2 by describing the two wave-models. It continues in §3 by investigating the evolution of both unidirectional and directional sea-states. Concluding remarks can be found in §4.

2 Wave Models

Two wave-models have been applied in order to model, and understand, the evolution of realistic directionally spread sea-states. The first is the fully-nonlinear numerical wave-model of Bateman *et al.* (2001). The second is the evolution-equation of Zakharov (1968). Whilst the former solves the exact fully-nonlinear boundary conditions it gives very little direct indication as to the physical mechanisms that govern the evolution of a sea-state. In contrast, Zakharov (1968) enables the interactions at each order to be separated and the dominant physical processes to be determined.

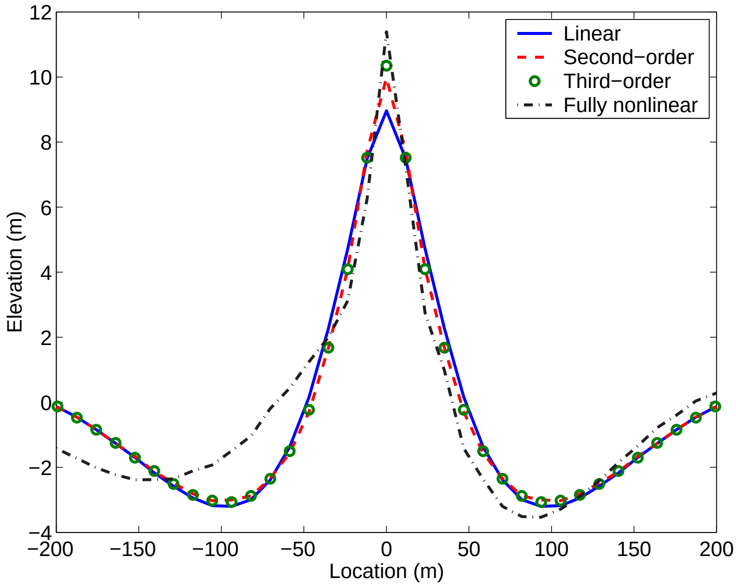
Bateman *et al.* (2001) is an extension to three-dimensions of the unidirectional wave-model of Craig & Sulem (1993). It is applied by time-marching the surface profile, $\eta(x, y)$, and the velocity potential on that surface, $\varphi(x, y)$, as first suggested by Zakharov (1968). At each time-step the horizontal derivatives of the surface, and of the velocity potential, can be rapidly calculated using fast Fourier transform techniques. However, the calculation of the vertical derivative requires the application of a Dirichlet-Neumann operator that transforms values of the potential on the surface into its vertical gradient. This is achieved by evaluating a high-order Taylor series. This last step is the reason for the efficiency of the Bateman *et al.* (2001) model; allowing realistic directional spectra, with a large range of frequencies, to be successfully modelled. In contrast, the wave-model of Johannessen & Swan (2003), itself an extension of the unidirectional wave-model of Fenton & Rienecker (1982), solved for the unknown coefficients using large matrix inversion. This is computationally intensive, leading to an inefficient model incapable of describing realistic directional spectra.

In contrast, Zakharov (1968) is an integro-differential equation that has been derived to fourth-order by Krasitskii (1994). At each order it is possible to isolate the *bound* and *resonant* interactions that occur, and hence, it is an excellent tool for understanding the physical mechanisms that govern the evolution of a wave-field. In this model the wave spectrum is time-marched according to which interactions are desired; making it possible to model the evolution of a wave-field including only the *bound* terms, only the *resonant* terms, or a mix of terms at different orders. This allows the dominant physical processes to be isolated. A detailed account of how it can be applied numerically is given by Annenkov & Shrira (2001), and a comparison with laboratory data is discussed in Shemer *et al.* (2001).

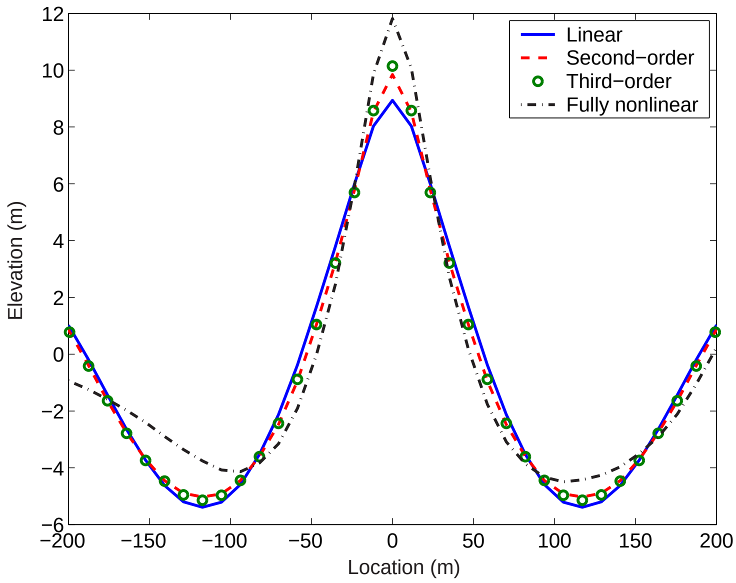
In order to investigate the evolution of large focussed wave-events the wave-models have been initiated with a specified wave spectrum at one point in time, $t = t_0$, when the sea-state is dispersed. The phasing of the wave components has been determined on the basis of the linear dispersion relationship such that a perfectly focussed wave-event will occur at $t = 0s$. The wave-models are then time-marched up to and beyond the largest crest elevation and the changes to the wave profile and the wave spectrum ascertained. In the Zakharov (1968) model the spectrum at each time-step is given in terms of the underlying linear wave components. In contrast, the Bateman *et al.* (2001) model gives the spectrum of the surface profile, which inevitably includes *bound* terms. However, it is possible to remove some of these *bound* terms by running the same cases so that a focussed wave trough occurs rather than a focussed wave crest. The trough focussed wave profile is then subtracted from the crest focussed profile and the Fourier transform of this result is a wave spectrum from which all the even-order (for example, second- and fourth-order) terms have been removed. However, odd-order terms are still present, and, although small, can potentially contaminate the results.

3 Results

The fully-nonlinear model of Bateman *et al.* (2001) has been used to model the evolution of two unidirectional sea-states; both of which are characterised by a JONSWAP spectrum of peak period $T_p = 12.8s$. However, the two spectra differ in their degree of broad-bandedness: the first has a peak enhancement factor $\gamma = 1$ (case J1D0) and the second a peak enhancement factor $\gamma = 5$ (case J5D0). Figure 1 shows that in both cases the fully-nonlinear wave profile of the extreme wave-event has a crest elevation much larger than that predicted by the solution of Zakharov (1968) that includes only the *bound* terms. Therefore, the formation of a large wave in a unidirectional sea-state cannot be modelled without taking into account the evolution of the wave spectrum. Figure 2 shows the rapid evolution of the freely propagating wave components of cases J1D0 and J5D0. In both sea-states this analysis shows that the underlying linear spectrum broadens considerably during the evolution of an extreme wave-event. This broadening is associated with an increase in the amplitude sum of the wave spectrum (Figure 3), and hence, the broadening is responsible for the large increases in maximum crest elevation. The physical mechanisms responsible for this evolution can be determined by applying Zakharov (1968). Indeed, figure 4 shows that an excellent agreement between the two models can be found by including only the third-order *resonant* terms. Furthermore, the evolution of these spectra is occurring much more rapidly than is predicted in random sea-states by Hasselmann (1962). The strong correlation between the phase of the wave components allows the third-order *resonant* interactions to broaden the wave spectrum over the time-scale of tens of wave periods rather than hundreds. Whilst this is true for unidirectional sea-states, in more realistic directionally spread wave-fields these conclusions need revising.

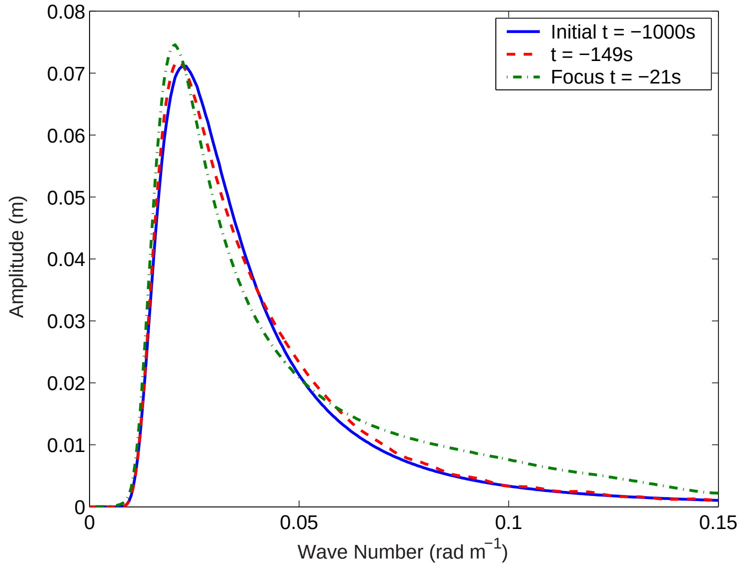


(a) J1D0

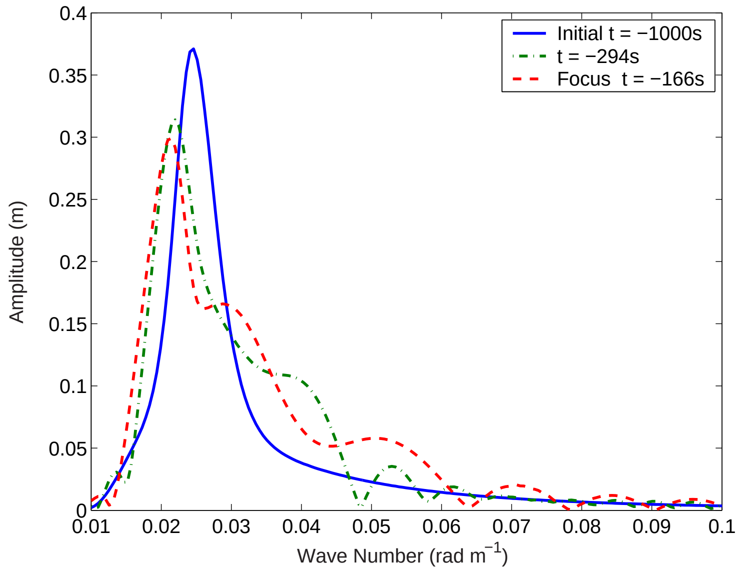


(b) J5D0

Fig. 1: Surface profile of extreme wave-events modelled using only the first-, second- and third-order *bound* terms of Zakharov (1968), and fully-nonlinearly using Bateman *et al.* (2001).



(a) J1D0



(b) J5D0

Fig. 2: The evolution of the underlying linear spectrum calculated using Bateman *et al.* (2001).

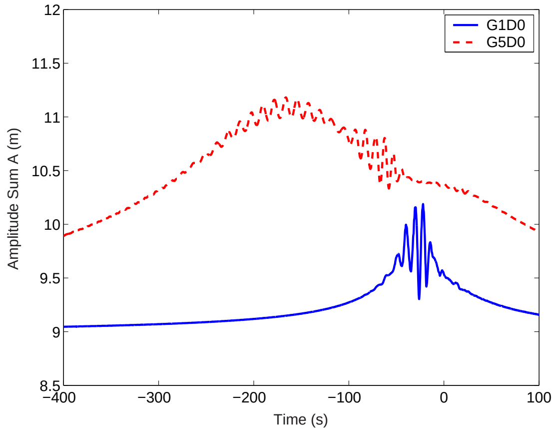
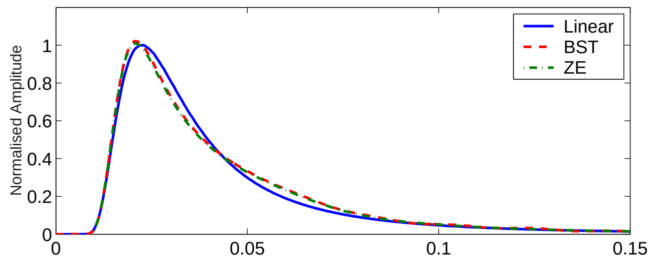


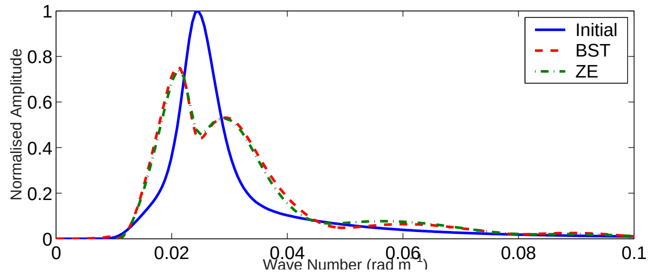
Fig. 3: The amplitude sum of the underlying linear spectrum. The high frequency oscillations represent third-order nonlinearites.

The fully-nonlinear model of Bateman *et al.* (2001) has also been used to model the evolution of two directional sea-states; both of which are characterised by a JONSWAP spectrum of peak period $T_p = 12.8s$ and peak-enhancement factor $\gamma = 5$. However, the two spectra differ in their degree of directional spreading: the first corresponds to a wrapped normal distribution with a standard deviation of $\sigma_\theta = 5°$ (case J5D5), while the second has a standard deviation $\sigma_\theta = 30°$ (case J5D30). Figure 5 shows that in both cases the crest elevation associated with the extreme wave profile is actually *lower* than that predicted by second-order theory. An analysis of the amplitude sum of the spectra (figure 6) indicates that the evolution of the amplitude spectra does not directly explain the reduction in crest elevation: in case J5D5 the amplitude sum increases from 9m to 11.6m; whilst, in case J5D30 it decreases from 9m to 8.7m. These changes to the amplitude sum are directly related to changes in the bandwidth of the two spectra, with a broadening of case J5D5 and a narrowing of J5D30. The actual changes to the spectral shape are considered in more detail in Gibson & Swan (2005). However, it is clear that there is a further factor to consider in determining the maximum crest elevation.

The maximum crest elevation associated with an extreme wave-event at any point in time, t_e, is the product of four inter-related factors. The first is the amplitude sum of the underlying linear wave spectrum at the initial time, $A_0 = A|_{t=t_0} = \sum_i^N a_i|_{t=t_0}$, where a_i are the amplitudes of the wave components. The second is the change to this amplitude sum due to *resonant* interactions ΔA. The third is the nonlinear correction due to *bound* interactions. Finally, the fourth is the phase relationship between the wave components. These factors are not independent; for example, the *bound* correction depends upon the amplitude sum of the wave components and their phasing. Despite this, it can be informative to investigate these factors in terms of a number of ratios:



(a) J1D0



(b) J5D0

Fig. 4: Comparison between the evolution of the underlying linear spectrum calculated using Bateman *et al.* (2001) with that calculated using Zakharov (1968). All the results are from $t = -100s$.

$$
\begin{aligned}
\eta_{actual} &= FA|_{t_0} = F_0F_1F_2A|_{t_0}\,, \text{where,} \\
F_0 &= (\Delta A + A_0)/A_0\,, \\
F_1 &= \eta_{bound}/(\Delta A + A_0)\,, \\
F_2 &= \eta_{actual}/\eta_{bound}\,,
\end{aligned} \tag{1}
$$

and η_{bound} defines the crest elevation, including the effect of *bound* interactions, if a perfectly focussed wave-event occurred, and η_{actual} is the actual crest elevation. Taking each in turn: F_0 is the increase in the amplitude sum of the underlying linear wave components; F_1 is the increase in the crest elevation due to *bound* interactions, if the wave components are perfectly focussed; and F_2 defines the focal quality of the event.

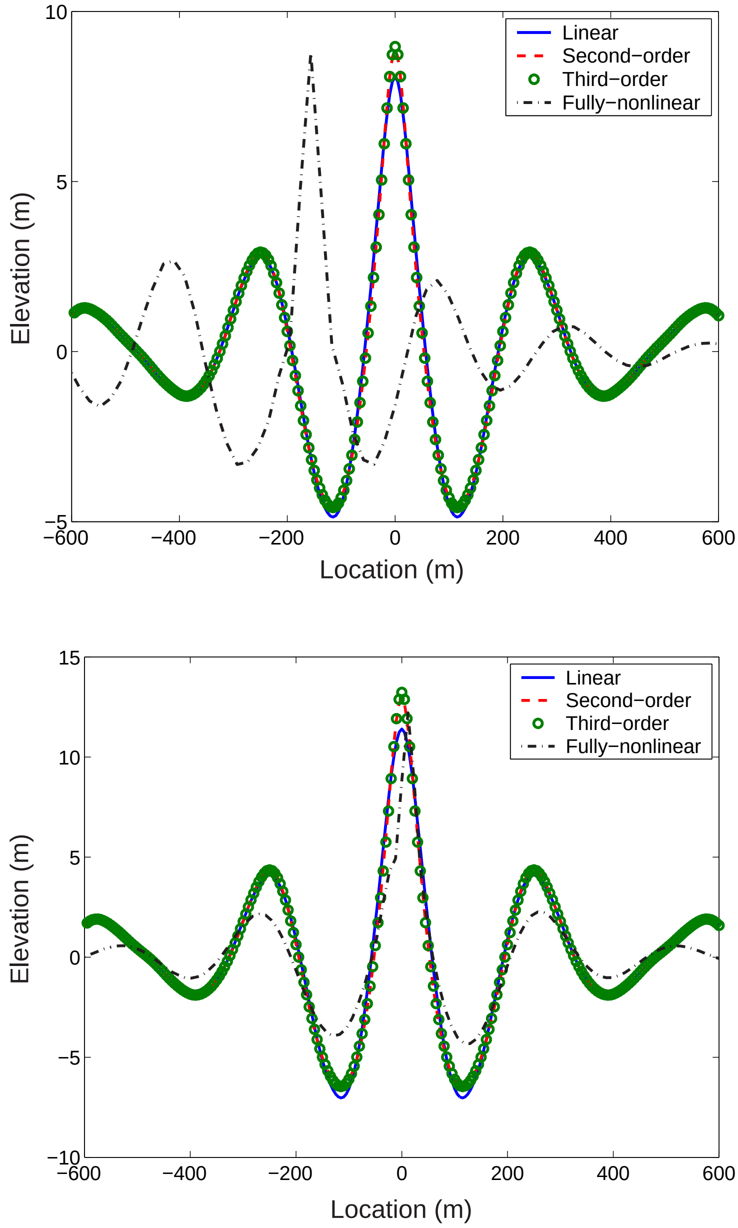


Fig. 5: Surface profile of extreme wave-events modelled using only the first-, second- and third-order *bound* terms of Zakharov (1968), and fully-nonlinearly using Bateman *et al.* (2001).

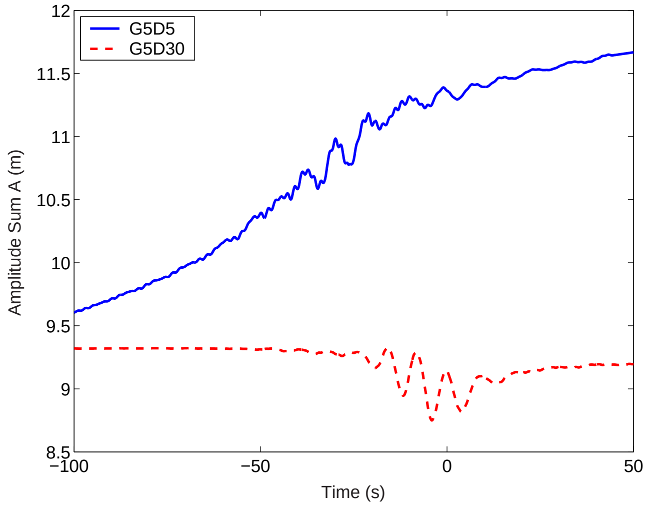


Fig. 6: The amplitude sum of the underlying linear spectrum. The high-frequency oscillations represent third-order nonlinearites, and hence, represent rapid changes to the steepness of the wave profile and should be ignored.

For example, if linear theory is considered then $F_0 = F_1 = F_2 = 1.0$ as there is no change to the amplitude sum of the spectrum, there are no *bound* interactions and the wave-event focusses perfectly. However, if second-order theory is considered then $F_0 = F_2 = 1.0$ as again there is no change to the amplitude sum, nor to the phasing; and, $F_1 = 1.12$ as the second-order *bound* interactions increase the crest elevation by 12% in case J5D0. Figure 7 shows the change in these factors as the initial amplitude sum of a unidirectional sea-state is increased. In all of the cases the initial phasing has been determined on the basis of the linear dispersion relationship and the *resonant* interactions have altered this phasing so that the event does not focus perfectly. However, the reduction in focal quality, F_2, is more than compensated by the increase in the amplitude sum of the wave spectrum, F_0.

In contrast, in directional sea-states, the focal quality is inversely correlated with increases in the amplitude sum of the spectrum, so that if $F_0 > 1$, $F_2 < 1$ (figure 8). Hence, the two balance and, for all but the most long-crested sea-states, the final crest elevation is less than that predicted by second-order theory. This highlights the fact that the result of the *resonant* interactions is critically dependent upon the phase relationship between the wave components: if the event focusses perfectly then the spectrum narrows and the amplitude sum of the underlying linear wave components reduces; however, if the event is imperfectly focussed then the spectrum can broaden, the amplitude sum increase, but the resulting crest elevation is limited by its poor focal quality. Therefore, in broad-banded directionally-spread sea-states it is not possible to obtain the large increases in crest elevation observed in similar unidirectional sea-states.

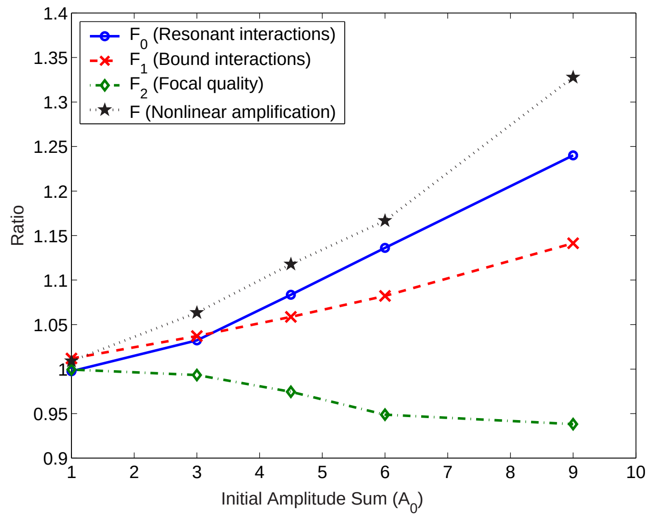


Fig. 7: The factors that effect the crest elevation of an extreme wave-event in a unidirectional sea-state.

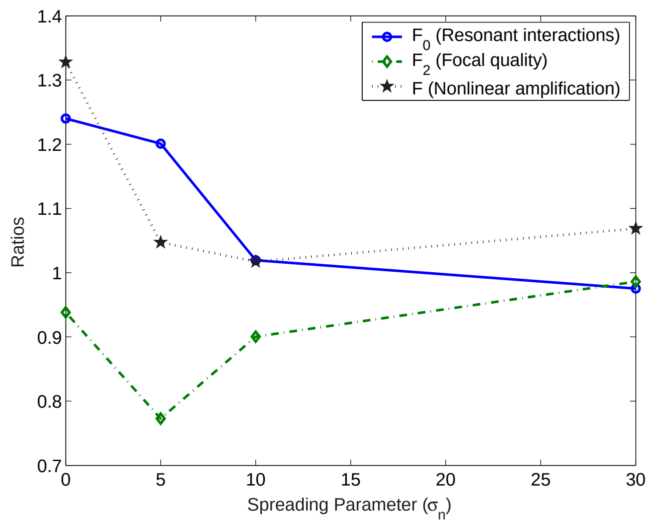


Fig. 8: The factors that effect the crest elevation of an extreme wave-event in a multi-directional sea-state characterised by a JONSWAP spectrum with a peak-enhancement factor $\gamma = 5.0$.

However, whilst it is not possible to obtain large increases in crest elevation if the spectrum is broad-banded, it is possible in more narrow-banded sea-states. Bateman *et al.* (2001) has been used to model the evolution of a Gaussian sea-state with a small directional spread of $\sigma_n = 5°$, characteristic of swell waves. In this case the sea-state disperses slowly, and the balance between changes to the spectral bandwidth due to *resonant* interactions and the focal quality of the extreme-event is balanced towards the former. Figure 9 shows that the maximum crest elevation is much larger than that predicted by a *bound* wave solution. It therefore follows that, swell-dominated sea-states are perhaps those in which *rogue* waves are intrinsically more likely to occur.

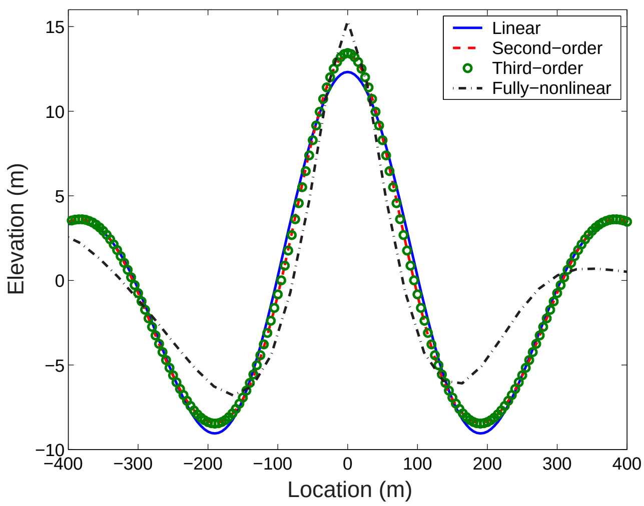


Fig. 9: Surface profile of an extreme wave-event in a swell-dominated sea-state, characterised by a Gaussian spectrum with a spreading parameter $\sigma_n = 5°$.

4 Concluding Remarks

The fully-nonlinear evolution of both unidirectional and directional sea-states has been investigated. It has been shown that in the former third-order *resonant* interactions lead to the rapid evolution of the wave spectrum, and to an increase in the crest elevation of an extreme wave-event. Whereas, in the latter the extreme crest-elevation is less than that predicted by second-order theory. This reduction is due to the balance between the phasing of the wave components and the spectral changes that occur during the formation of a large event. This suggests that the focussing of wave components is not a possible mechanism by which *rogue* waves can form in broad-banded sea-states. However, in more narrow-banded Gaussian spectra, characteristic of swell-dominated sea-states, large waves are slow to disperse. Hence, the balance is now in favour of the *resonant* interactions and large increases in crest elevation can be obtained.

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