Dependence of Freak Wave Occurrence on Kurtosis

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Abstract. The occurrence probability of freak waves is formulated as a function of number of waves and surface elevation kurtosis based on the weakly non-Gaussian theory. Finite kurtosis gives rise to a significant enhancement of freak wave generation. For fixed number of waves, the estimated amplification ratio of freak wave occurrence due to the deviation from the Gaussian theory is 50%-300%.

1 Introduction

The last decade freak waves have become an important topic in engineering and science. Freak wave studies started in the late 80's [1] and the high-order nonlinear effects on freak waves were discussed in the early 90's [2, 3]. Due to many research efforts, the occurrence of freak waves, their mechanism and detailed dynamical properties are now becoming clear [4, 5, 6, 7, 8]). It was concluded that the third order nonlinear interactions enhance the freak wave occurrence and are the primary cause of freak wave generation in a general wave field except for the case of strong wave-current interaction or wave diffraction behind an island [9].

Numerical and experimental studies have demonstrated that freak-like waves can be generated frequently in a two-dimensional wave flume without current, refraction or diffraction [2, 4, 6, 10]. Moreover, the numerical studies clearly indicate that a freak wave having a single, steep crest can be generated by the third order nonlinear interactions in deep-water [2]. Also, the theoretical background of freak wave generation has become more clear [5], but the quantitative occurrence probabilities on the ocean surface remain uncertain. In addition, it is still questionable how to characterize the dominant statistical properties of freak wave occurrence in terms of nonlinear parameters, spectral shape, water depth and so on.

Recently, Janssen [11] investigated the freak wave occurrence as a consequence of four-wave interactions including the effects of non-resonant four-wave interactions. He formulated the analytical relationship between the spectral shape and the kurtosis of the surface elevation. These results have the potential to unify the previous freak wave studies covering nonlinear interactions, spectral profiles, nonlinear statistics, etc.

The purpose of this study is to investigate the relationship between kurtosis and the occurrence probability of freak wave in a unidirectional wave train. First, the wave height distribution is formulated as a simple function of kurtosis by the non-Gaussian theory. Second, the maximum wave height distribution is obtained from the wave height distribution as a function of kurtosis and number of waves. Finally, the dependence of the occurrence probabilities of freak waves on kurtosis and the number of waves will be analyzed and discussed.

2 Theoretical Formulation of Freak Wave Occurrence

Following a central limit theorem, linear, dispersive random waves have a Gaussian probability distribution function (PDF) for the surface elevation. Finite amplitude effects result, however, in deviations from the Normal distribution, as measured by a finite skewness and kurtosis. For narrow band wave trains it will be shown that the wave height distribution only depends on the kurtosis. Therefore, we shall formulate the relationship between wave height distribution and kurtosis to examine analytically the effects of kurtosis on freak wave occurrence.

We assume that waves to be analyzed here are unidirectional with narrow banded spectra and satisfy the stationary and ergodic hypothesis. Let $\eta(t)$ be the sea surface elevation as a function of time t and $\zeta(t)$ an auxiliary variable such that $\eta(t)$ and $\zeta(t)$ are not correlated. Assuming both $\eta(t)$ and $\zeta(t)$ are real zero-mean functions, we have

$$Z(t) = \eta(t) + i\zeta(t) = A(t)e^{i\phi(t)}, \tag{1}$$

$$A(t) = \sqrt{\eta^2(t) + \zeta^2(t)}, \tag{2}$$

$$\phi(t) = \tan^{-1}\left(\frac{\zeta(t)}{\eta(t)}\right), \tag{3}$$

where A is the envelope of the wave train and ϕ the phase. For weakly nonlinear waves deviations from the Normal distribution are small. In those circumstances the PDF of the surface elevation can be described by the Edgeworth distribution, and the joint probability density function of η and ζ is known. The PDF of the envelope A now follows immediately from an integration of the joint probability distribution over the phase ϕ. It is then found that the first term gives the Rayleigh distribution, while the terms involving the skewness and etc, all integrate to zero because they are odd functions of the phase. The third term does give contributions to the probability distribution for the envelope and as a result we find

$$p(A) = Ae^{-\frac{1}{2}A^2}\left[1 + \frac{1}{4}\left(\kappa_{40} + \kappa_{22}\right)\left(1 - A^2 + \frac{1}{8}A^4\right)\right], \tag{4}$$

where κ_{ij} is joint cumulant between η and ζ and we have used $\kappa_{40} = \kappa_{04}$, a relation that can easily be verified. Following Mori and Janssen [12], $\kappa_{22} = \kappa_{40}/3$ and the final result for the narrow-band approximation of the PDF of the envelope becomes

$$p(A) = Ae^{-\frac{1}{2}A^2}\left[1 + \frac{1}{3}\kappa_{40}\left(1 - A^2 + \frac{1}{8}A^4\right)\right]. \quad (5)$$

From this result interesting consequences on the distribution of maximum wave heights may be obtained. In the narrow band approximation wave height H equals $2A$ and hence the wave height PDF becomes

$$p(H) = \frac{1}{4}He^{-\frac{1}{8}H^2}\left[1 + \kappa_{40}A_H(H)\right], \quad (6)$$

where $A_H(H) = \frac{1}{384}\left(H^4 - 32H^2 + 128\right)$. The exceedance probability $P_H(H)$ for wave height then follows from an integration of the wave height PDF from H to ∞:

$$P_H(H) = e^{-\frac{1}{8}H^2}\left[1 + \kappa_{40}B_H(H)\right], \quad (7)$$

where $B_H(H) = \frac{1}{384}H^2\left(H^2 - 16\right)$.

We adopt a simple freak wave definition. A freak wave is thought to occur when the maximum wave height H_{max} exceeds twice the significant wave height $H_{1/3}$ of the wave train. The PDF of maximum wave height p_m in wave trains can be obtained once the PDF of wave height $p(H)$ and exceedance probability of wave height $P(H)$ is known [13], thus,

$$p_m(H_{max})dH_{max} = N_0[1 - P(H_{max})]^{N_0 - 1}p(H_{max})dH_{max}. \quad (8)$$

Substitution of Eqs.(6) and (7) into Eq.(8), gives the PDF of the maximum wave height, p_m,

$$p_m(H_{max})dH_{max} = \frac{N}{4}H_{max}\,\mathrm{e}^{-\frac{H_{max}^2}{8}}\left[1 + \kappa_{40}A_H(H_{max})\right] \times \exp\left\{-N\mathrm{e}^{-\frac{H_{max}^2}{8}}\left[1 + \kappa_{40}B_H(H_{max})\right]\right\}dH_{max}, \quad (9)$$

and exceedance probability of the maximum wave height P_m,

$$P_m(H_{max}) = 1 - \exp\left\{-N\mathrm{e}^{-\frac{H_{max}^2}{8}}\left[1 + \kappa_{40}B_H(H_{max})\right]\right\}. \quad (10)$$

Eq.(9) is now evaluated as a function of N and κ_{40} (or μ_4). For $\kappa_{40} = 0$ results are identical to the ones following from the Rayleigh distribution. For simplicity it will be assumed that $H_{1/3} = 4\eta_{rms}$ (with η_{rms} the square root of the mean of the surface elevation variance). The freak wave condition therefore becomes $H_{max}/\eta_{rms} \geq 8$, and we obtain from Eq.(10) the following simple formula to predict the occurrence probability of the freak wave as a function of N and κ_{40},

$$P_{freak} = 1 - \exp\left[-\alpha N(1 + 8\kappa_{40})\right] \quad (11)$$

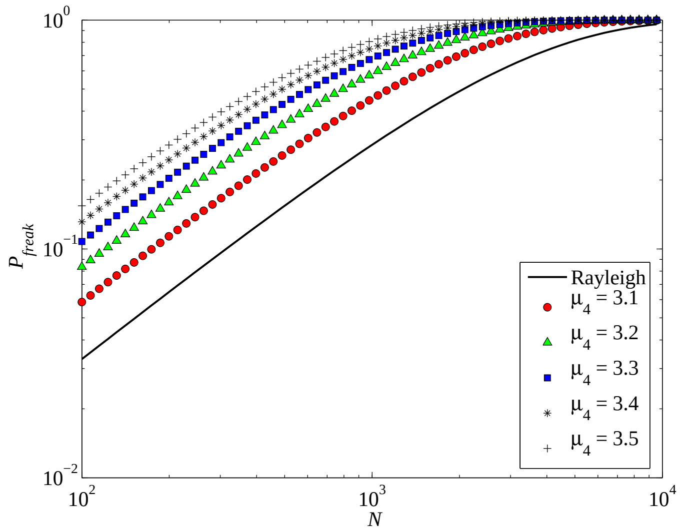

Fig. 1. Occurrence probability of freak wave as a function of number of waves N and kurtosis μ_4.

where $\alpha = \mathrm{e}^{-8}$.

Figure 1 shows for increasing μ_4 from 3.0 to 3.5 the comparison between the linear (Rayleigh) theory and present theory of the occurrence probability of the freak wave, P_{freak}, as a function of the number of waves N. For the case of N=100, the occurrence probability of the freak wave predicted by the linear theory is 3.3%, while it is 15.4% according to Eq.(9) with μ_4=3.5, and for the case of N=1000, the occurrence probability of the freak wave is 28.5% by the linear theory, while it is 81.3% according to Eq.(9) with μ_4=3.5. Thus, freak waves in a strong nonlinear field can occur several times more frequently than in a linear wave field, which obeys the Rayleigh distribution.

Figure 2 shows the ratio of freak wave occurrence probability R_{freak} predicted by the present theory and the Rayleigh theory, as a function of kurtosis μ_4. For the case of a small number of waves $N \leq 250$, the ratio R_{freak} linearly depends on μ_4. If μ_4 is 3.1 and $N \leq 500$, the occurrence probability of freak waves is 50% (two times) more than according to linear theory. On the other hand, the increment of R_{freak} decreases as the number of waves increases. This is because for very large number of waves even in linear theory the maximum wave height almost always exceeds $2 \times H_{1/3}$.

3 Conclusion

In this study, the kurtosis and related high-order cumulants were evaluated on the basis of Janssen's work [11]. Second, the wave height and the maximum wave

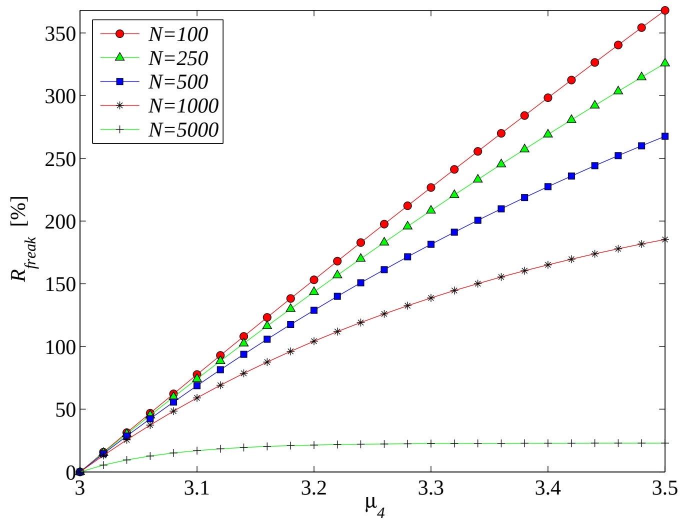

Fig. 2. Ratio of freak wave occurrence predicted by Eq.(11) to the Rayleigh theory.

height distribution were formulated as a function of kurtosis. Finally, the freak wave occurrence probability was formulated as a function of kurtosis and number of waves. From the theoretical frame work, we have the following remarks.

- The second order cross-cumulant between surface elevation and its envelope, κ_{22}, is 1/3 of the fourth cumulant, κ_{40}, of the surface elevation.
- The weakly non-Gaussian theory shows the dependence of the maximum wave height distribution on kurtosis.
- The occurrence probability of freak waves is significantly enhanced by the kurtosis increase as a consequent of the wave-wave interactions.

In order to check the validity of the theories developed here, systematic and continuous field measurement of freak waves will be critically required.

References

[1] R.G. Dean. Freak waves: a possible explanation. In A. Tørum and O.T. Gudmestad, editors, *Water wave kinematics*, pages 609–612. Kluwer Academic Pub., 1990.
[2] T. Yasuda, N. Mori, and K. Ito. Freak waves in a unidirectional wave train and their kinematics. In *Proc. of the 23th Int. Conf. on Coastal Eng.*, volume 1, pages 751–764, Venice, 1992. ASCE.
[3] T. Yasuda and N. Mori. High order nonlinear effects on deep-water random wave trains. In *International Symposium: Waves-Physical and Numerical Modelling*, volume 2, pages 823–332, Vancouver, 1994.
[4] K. Trulsen and K. Dysthe. Freak waves-a three-dimensional wave simulation. In *Proceedings of the 21st Symposium on Naval Hydrodynamics*, pages 550–558. (National Academy Press, 1997.

[5] A.R. Osborne, M. Onorato, M. Serio, and S. Bertone. The nonlinear dynamics of rogue waves and holes in deep water gravity wave trains. *Physical Letter A*, (275):386–393, 2000.

[6] Onorato M., A.R. Osborne, M. Serio, and S. Bertone. Freak waves in random oceanic sea states. *Physical Review Letter*, (86):5831–5834, 2001.

[7] Sverre Haver. Evidences of the existence of freak waves. In M. Olagnon and G. Atanassoulis, editors, *Rogue Waves*, pages 129–140. 2001.

[8] N. Mori, P. Liu, and T. Yasuda. Analysis of freak wave measurements in the Sea of Japan. *Ocean Engineering*, 29(11):1399–1414, 2002.

[9] M. Olagnon and G. Athanassoulis, editors. *Rogue Waves*. IFRMER, 2000.

[10] C.T. Stansberg. Extreme waves in laboratory generated irregular wave trains. In A. Tørum and O.T. Gudmestad, editors, *Water wave kinematics*, pages 573–590. Kluwer Academic Pub., 1990.

[11] P.A.E.M. Janssen. Nonlinear four-wave interactions and freak waves. *J. Phys. Oceanogr.*, (33):2001–2018, 2003.

[12] N. Mori and P.A.E.M Janssen. On kurtosis and occurrence probability of freak waves. *in preparation*, 2005.

[13] Y. Goda. *Random Seas and Design of Maritime Structures.* World Scientific, 2nd edition, 2000.