Quasi-Resonant Interactions and Non-Gaussian Statistics in long crested waves

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Abstract. We compare the statistical properties of long crested surface gravity waves recorded in a long wave tank with numerical results obtained from a modified kinetic equation derived from the Dysthe equation. We find experimentally and theoretically that the statistical properties of the surface elevation depend on the ratio between the steepness and the spectral band-width of the spectrum at the wave maker. We compare succesfully the kurtosis computed from the experimental data with the one obtained by the statistical description of the Dysthe equation.

1 Introduction

In the standard weak turbulence theory for surface gravity waves an irreversible transfer of energy between free modes is known to be allowed only if the following resonant conditions are satisfied:

$$\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 = 0 \tag{1}$$

$$\omega_1 + \omega_2 - \omega_3 - \omega_4 = 0, \tag{2}$$

where $\mathbf{k}_i$ are wave vectors and ω_i are angular frequencies related to wave number through the dispersion relation. For surface gravity waves in deep water this is a very fundamental result, obtained more than 35 years ago independently by Hasselmann [1] and Zakharov [2]. Starting from the fully nonlinear inviscid equations for surface gravity waves, under a number of assumptions, they were able to derived the so called *kinetic equation* that describes the evolution in time of the wave spectrum. The kinetic equation is of fundamental importance for applicative purposes because it is the base for operational wave models: in order to forecast waves, an approximate version of this kinetic equation is daily numerically solved in many research centers (it should be here mentioned that besides the nonlinear interactions, source terms such as wave breaking and wave forcing by the wind must be included in the energy balance equation).

More recently, it has been found that the modulational instability can be considered as an important mechanism for the formation of large amplitude waves [3],[4],[5]; for random waves the instability is particularly relevant for long crested wave packets, see [6], [7]. In spectral space, the modulational instability can be thought as a four wave interaction process which has the peculiarity of being *quasi-resonant*, i.e. equation (2) is not exactly satisfied, but $|\omega_1+\omega_2-\omega_3-\omega_4| \leq O(\epsilon^2)$. Because of this, the standard kinetic equation cannot in principle describe the dynamics of the modulational instability. Nevertheless, P. Jannsen in [8] derived a *kinetic equation* which includes *quasi-resonant interaction* and therefore it is capable of describing the statistical properties of groups of random waves that can become unstable. In [8] a formula for the kurtosis which depends on the wave spectrum of the free waves is also derived.

Here we test these ideas and compare the prediction of the just mentioned *quasi-resonant kinetic equation* with recent experiments performed in a 270 meters wave flume in Norway (Marintek facility) where different JONSWAP spectra, characterized by different steepness and spectral band-width, have been considered as boundary conditions at the wave-maker. Our analysis is concentrated on the evolution of the kurtosis along the tank. In order to compare our experimental results with the theory an evolution equation in space is needed. More in particular, we will build a kinetic equation including four wave quasi-resonant interactions starting from the Dysthe equation written as an evolution in space. The paper is organized as follows: in Section 2, following the approach in [8], we derive from the Dysthe equation the kinetic equation including the quasi-resonant interactions. In Section 3 experimental details are given. Finally experimental and numerical results will be compared in Section 4.

2 Evolution in space of the quasi resonant Kinetic equation

Our aim is here to estimate the kurtosis from the spectral properties of the Dysthe envelope equation (an extension of the nonlinear Schreodinger equation). The approach is the same as the one used by Janssen [8] to derive a kinetic equation from the Zakharov equation [9]; here the only complication is that we are dealing with a boundary-value problem, therefore the kinetic equation should be written as an evolution equation in space rather than in time. Moreover the hypothesis of homogeneity does not hold anymore (as we will see, the statistical properties of the surface elevation changes along the tank) and will be substituted by the hypothesis of stationarity. Here we will just give a short description of the procedure (the interested reader should read [8]).

In order to compare our experimental data with the theory we consider the Dysthe equation written as an evolution equation in space (see for example [10]):

$$\frac{\partial B}{\partial x} + i\frac{k_0}{\omega_0^2}\frac{\partial^2 B}{\partial t^2} + ik_0^3|B|^2B - \frac{8k_0^3}{\omega_0}|B|^2\frac{\partial B}{\partial t} +$$

$$-\frac{2k_0^3}{\omega_0}B^2\frac{\partial B^*}{\partial t} - i\frac{4k_0^3}{\omega_0^2}B\frac{\partial\overline{\phi}}{\partial t}\bigg|_{z=0} = 0, \quad at \quad z=0 \tag{3}$$

$$\frac{\partial\overline{\phi}}{\partial z} = -k_0\frac{\partial|B|^2}{\partial t}, \qquad at \quad z=0 \tag{4}$$

$$\nabla^2\overline{\phi} = 0, \qquad for \quad -\infty < z < 0, \tag{5}$$

$$\frac{\partial\overline{\phi}}{\partial z} = 0, \qquad at \quad z=-\infty \tag{6}$$

x is space, t is time, ω_0 and k_0 are respectively the dominant angular frequency and wave-number. B is the complex wave envelope related to the surface elevation η as follows:

$$\eta(x,t) = \bar{\eta} + \frac{1}{2}\left(B(x,t)e^{i(k_0x-\omega_0t)} + B^{(2)}(x,t)e^{i2(k_0x-\omega_0t)} + ... + c.c.\right), \tag{7}$$

where $c.c.$ denotes complex conjugate; $\bar{\eta}$ and $B^{(2)}$ are functions of B. This implies that the higher harmonics (and the zero harmonic) are phase locked to the complex envelope B that describes the evolution of free waves. Their explicit forms are given for example in [10]. In order to relate the kurtosis to the spectral properties of the surface elevation, it is useful to write all the equations in frequency Fourier space. Applying the Fourier transform to the set of equations (3-6), we obtain, as expected, the following Dysthe equation:

$$\frac{\partial B_1}{\partial x} - ikB_1 = i\int D_{1,2,3,4}B_2^*B_3B_4\delta_{12}^{34}d\omega_{2,3,4}, \tag{8}$$

where $D_{1,2,3,4} = D(\omega_1,\omega_2,\omega_3,\omega_4)$ is the coupling coefficient (its analytical form is given in [11]), $B_i = B(\omega_i)$ and δ_{12}^{34} is just a short notation for $\delta(\omega_1+\omega_2-\omega_3-\omega_4)$. We now consider the contribution to the kurtosis only from free waves (we do not consider the effect of B_2 and $\overline{\eta}$). Using (7), it is straightforward to obtain:

$$kurt = \frac{<\eta^4>}{<\eta^2>^2} = \frac{3}{8<\eta^2>^2}\int <B_1B_2B_3^*B_4^*> \delta_{12}^{34}d\omega_{1,2,3,4}, \tag{9}$$

where brackets $< ... >$ stand for ensemble averages. Now we assume that waves are weakly nonlinear (this is also an hypothesis needed for deriving the Dysthe equation) and we split the fourth order correlator as

$$<B_1B_2B_3^*B_4^*> = <B_1B_3^*><B_2B_4^*> + <B_1B_4^*><B_2B_4^*> + C_{1,2,3,4} \tag{10}$$

where $C_{1,2,3,4}$ is the fourth order cumulant which is exactly zero for a Gaussian random process. Using equation (8), we can calculate the evolution of the cumulat $C_{1,2,3,4}$ by making the hypothesis of stationarity, $<B_iB_j^*> = N_i\delta(i-j)$. Assuming that the process is Gaussian at $x=0$ (this is the condition imposed by us at the wave maker), we obtain the following final form for the kurtosis:

$$kurt = 3 + \frac{6}{<\eta^2>^2}\int D_{1,2,3,4}N_1N_2N_3\frac{1-Cos(\Delta kx)}{\Delta k}\delta_{12}^{34}d\omega_{1,2,3,4} \tag{11}$$

where $\Delta k = k_3 + k_4 - k_1 - k_2$. An analogous equation for the evolution of the kurtosis in time has been obtained in [8]. In order to compute numerically the evolution of the kurtosis, the evolution of N is needed, therefore, using the standard procedure, we obtain from the deterministic Dysthe equation the following *quasi-resonant kinetic equation*:

$$\frac{\partial N_1}{\partial x} = 4 \int |D_{1,2,3,4}|^2 \frac{Sin(\Delta k x)}{\Delta k} (N_3 N_4 (N_1 + N_2) - N_1 N_2 (N_3 + N_4)) \delta_{12}^{34} d\omega_{2,3,4}$$

Our goal is to verify experimentally the prediction from equation (11). Note that here we have only discussed the contribution to the kurtosis from the interaction of free modes. It should be mentioned that bound modes can also give a contribution that can be included in (11) by using the complete relation between the surface elevation η and the complex envelope B, see [11].

3 Experimental Set-up

The experiment was carried out in the long wave flume at Marintek (see [12] for details). The length of the tank is 270 m and its width is 10.5 m. The depth of the tank is 10 meters for the first 85 meters and then is reduced to 5 meters for the rest of the flume. The effect of the jump from 10 to 5 meters is insignificant for the 1.5 seconds waves considered here. A sloping beach is located at the far end of the tank opposite the wave maker. After half an hour of an irregular wave run with peak period of 1.5 seconds, the wave reflection was estimated to be less than 5%. The wave surface elevation was measured simultaneously by 19 probes placed at different locations along the flume (Figure 1). The sampling frequency for each probe was 40 Hz. JONSWAP random wave signals where synthesized as

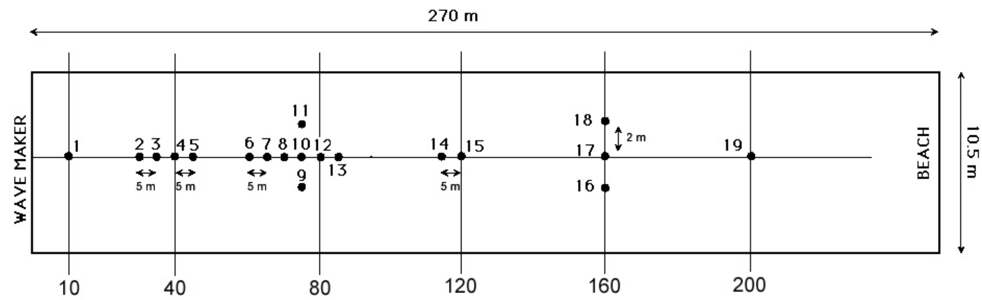

Fig. 1. Schematic of the wave tank facility at Marintek and location of wave probes.

sums of independent harmonic components, by means of the inverse Fast Fourier Transform of complex random Fourier amplitudes. Three different JONSWAP spectra with different values of α and γ have been investigated. All of them were characterized by a peak period of 1.5 seconds. In Table below we report the parameters that characterized each JONSWAP spectrum. In order to have

Experiment	γ	$H_s(m)$	$\epsilon = k_p H_s/2$	$\Delta f/f_p$	$\epsilon f_p/\Delta f$
I	6	0.16	0.15	0.08	1.87
II	3.3	0.14	0.13	0.09	1.44
III	1	0.11	0.1	0.28	0.36

Table 1. Parameters of the three different experiments performed at Marintek

sufficiently good statistics, a large number of waves was recorded. Note that the large amount of data is of fundamental importance for the convergence of higher order moments such as the kurtosis: for each type of spectrum, 5 different realizations with different sets of random phases have been performed. The duration of each realization was 32 minutes.

4 Results and conclusions

As mentioned before, we are mainly interested in the evolution of the kurtosis. We recall that for a Gaussian distribution the value of the kurtosis is 3, while

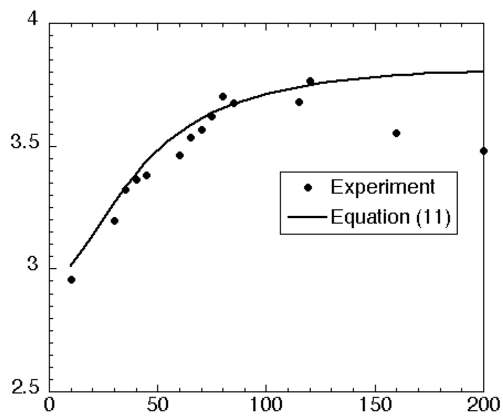

Fig. 2. Evolution of the kurtosis from filtered data for experiment I (see Table 1) and from equation (11).

larger values of kurtosis in a measured time series can give an indication of the presence of extreme events. Deviation from gaussian behaviour of the surface

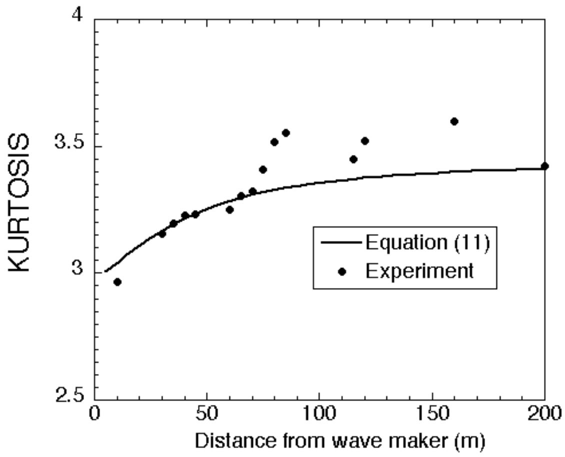

Fig. 3. Evolution of the kurtosis from filtered data for experiment II (see Table 1) and from equation (11).

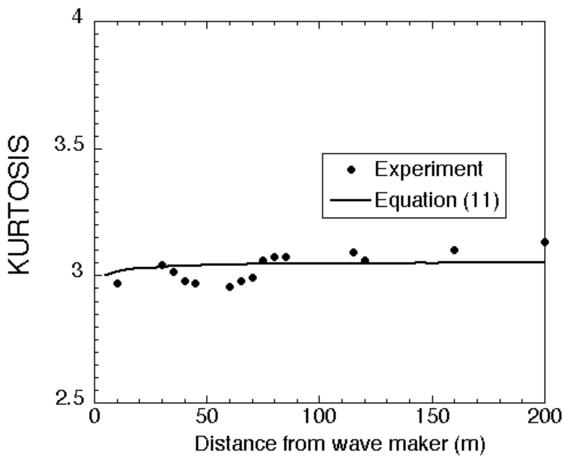

Fig. 4. Evolution of the kurtosis from filtered data for experiment III (see Table 1) and from equation (11).

elevation can possibly be associated to the Stokes contribution, i.e. waves that are phase locked to the peak of the wave spectrum: if ω_p is the dominant angular frequency in the spectrum, then we expect bound modes at $\omega = n\omega_p$ for $n = 2, 3, ...$ On the other hand the modulational instability is a typical process that takes place between free modes mainly near the peak of the spectrum, therefore close to $\omega = \omega_p$. In order to compare the theory developed (equation 11 is obtained considering only free modes) with experimental data, the free waves should be extracted from the time series. This is not an easy task, therefore here we simply filter our data by excluding all the frequencies larger than 1.1 Hz. By doing this we are confident that our filtered data do not have any contribution from the phase locked second, third and so on harmonics. In other words we build new time series in which the Stokes contribution has been removed. The second step consists in calculating the kurtosis on the filtered data and in comparing it with the kurtosis of the original data. In Figures 2- 4 we show the evolution along the tank of the kurtosis for the three experiments compared with equation (11). For the first two experiments, kurtosis grows quite rapidly and reaches its maximum between 25 and 30 wave lengths from the wave maker. For the third experiment, the kurtosis remains very close to the gaussian value. The qualitative behaviour of all three experiment is well captured by the theory previously described. Our figure 4 is also consistent with preliminary results obtained for the unfiltered kurtosis for experiment I in [13]. The results suggests a significant dependence of the statistical properties of surface gravity waves on the spectral shape of the initial spectrum. In particular, as it has been discussed in other papers ([4], [8]), the so called Benjamin Feir Index, given by the ratio between the steepness and the spectral band-width, plays an important role for determining the statistical properties of long crested waves. Although the comparison between theory and experiment can be considered as quite successful, we are unable to reproduce "exactly" the experimental data (see figure 4 where at the last probe the theory overestimates substantially the value of the kurtosis). Nevertheless, it should be mentioned that the model contains only the weakly nonlinear interactions in one dimensions, therefore three dimensional instabilities or wave breaking are not considered. Here we recall that any second order theory would give a very small contribution to the kurtosis with respect to the observed experimental data, [11]. As a final conclusion, we may state with some confidence that the BFI plays an important role for determining the statistical properties of surface elevation for the present experiment. We should recall here that the Benjamin-Feir-Index has been derived for long crested, narrow banded waves in deep water [4],[8], therefore it should be used with care for any other physical condition.

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